6.2 – Angle and Orthogonality in Inner Product Spaces

Definition: The angle $\boldsymbol{\theta}$ **between vectors u and v** in a real inner product space V is $\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$.

Example Find the cosine of the angle between A and B with respect to the standard inner product on M_{22}.
$A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

Theorem 6.2.1 Cauchy-Schwarz Inequality (generalization of Theorem 3.2.4)
If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, then $|\langle \mathbf{u},\ \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

Theorem 6.2.2 Triangle inequalities (generalization of Theorem 3.2.5)
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a real inner product space V, then:
a. $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)
b. $d(\mathbf{u},\ \mathbf{v}) \le d(\mathbf{u},\ \mathbf{w}) + d(\mathbf{w},\ \mathbf{v})$ (triangle inequality for distances)

Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a real inner product space V are **orthogonal** if $\langle \mathbf{u},\ \mathbf{v} \rangle = 0$.

#10 Show that the vectors are orthogonal with respect to the standard inner product on P_2.
$\mathbf{p} = 2 - 3x + x^2$, $\mathbf{q} = 4 + 2x - 2x^2$

#33 Let $C[-1,\ 1]$ have the integral inner product $\langle \mathbf{p},\ \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$ and let $\mathbf{p} = p(x) = x^2 - x$ and $\mathbf{q} = q(x) = x + 1$.
a. Find $\langle \mathbf{p},\ \mathbf{q} \rangle$.
b. Find $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$.

Explore: What is $\|\mathbf{p} + \mathbf{q}\|^2$? What about $\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$?

Theorem 6.2.3 Generalized Theorem of Pythagoras
If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a real inner product space, then
$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Example: Let $\mathbf{p} = p(x) = 1 - 2x^2$, $\mathbf{q} = q(x) = 4 - 2x + x^2$, $\mathbf{r} = r(x) = x + 2x^2$ and let P_2 have the standard inner product.

a. Compute the following: $\langle \mathbf{p}, \mathbf{q} \rangle$ and $\langle \mathbf{q}, \mathbf{r} \rangle$.

b. Now let P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + k a_2 b_2$. Find k so that $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.

c. Using this new inner product, compute $\langle \mathbf{q}, \mathbf{r} \rangle$.

#17 Do there exist scalars k and l such that the vectors $\mathbf{p}_1 = 2 + kx + 6x^2$, $\mathbf{p}_2 = l + 5x + 3x^2$, $\mathbf{p}_3 = 1 + 2x + 3x^2$ are mutually orthogonal with respect to the standard inner product on P_2?

Definition: (inner product space analog of Definition 2 in Section 4.9) If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$ (pronounced "W perp").

Theorem 6.2.4 (generalization of Theorem 4.9.6 (a) and (b))
If W is a subspace of a real inner product space V, then:
a) $W^\perp$ is a subspace of V.
b) $W \cap W^\perp = \{\mathbf{0}\}$.

Theorem 6.2.5 (generalization of Theorem 4.9.6(c))
If W is a subspace of a real finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W, that is, $(W^{\perp})^{\perp} = W$.

#28 Find a basis for the orthogonal complement of the subspace of R^5 spanned by the vectors.
$\mathbf{v}_1 = (1, 4, 5, 6, 9), \mathbf{v}_2 = (3, -2, 1, 4, -1), \mathbf{v}_3 = (-1, 0, -1, -2, -1), \mathbf{v}_4 = (2, 3, 5, 7, 8)$